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Division of Any Two Triangular Shape Fuzzy Numbers via Mathematica

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ABSTRACT *: This paper gives the result of the division of any two triangular shape fuzzy numbers using mathematica. We know that, usually, the division of two triangular shape fuzzy numbers does not provide the triangular shape fuzzy number. This is also illustrated in this paper. For this reason some examples have been considered here for verifying the proposed mathematica program. Computations were carried out by mathematica.*

Keywords: *Fuzzy number, fuzzy arithmetic operations, α-cut, Mathematica.*

1. INTRODUCTION :

Twenty five years ago, after the forty years old introduction of the fuzzy sets by Zadeh (1965), Dubois and Prade (1980, 1988) stated the exact analytical fuzzy mathematics and introduced the well known LR model and the corresponding formulas for the fuzzy operations. Since the introduction of the extension principle by Zadeh (1965), the arithmetic of fuzzy numbers has gained importance both from the theoretical and the practical points of view. In general, the arithmetic operations on fuzzy numbers can be approached by the direct use of the membership function by the Zadeh (1965) extension principle or by the equivalent use of α-cuts representation introduced by Goetschel and Voxman (1986). The basic arithmetic structure for fuzzy numbers was developed by Mizumoto and Tanaka (1976, 1979), Nahmias (1978) and others.

2. FUZZY SET :

Let X be a nonempty set. A fuzzy set A in X is characterized by its membership function $A: X \rightarrow [0,1]$ and A(x) is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$. The value zero is used to represent complete non-membership, the value one is used to represent complete membership, and values in between are used to represent intermediate degrees of membership. The mapping A is also called the membership function of fuzzy set A.

Example:
The membership function of the fuzzy set of real numbers "close to one" can be defined as

$$A(x) = exp\left(-\beta(x-1)^2\right),$$

where β is a positive real number.

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3. FUZZY NUMBER

A fuzzy number is one which is described in terms of a number word and a linguistic modifier, such as approximately, nearly, or around. The concept can be captured by a fuzzy set defined on the set of real numbers. Its membership function should assign the degree of 1 to the central value and degrees to other numbers that reflect their proximity to the central value according to some rule. The membership function should thus decrease from 1 to 0 on both sides of the central value. Fuzzy sets of this kind are called fuzzy numbers.

General form of a fuzzy number:

$$A(x)=\begin{cases} 0 & for\, x \le a, \\ f(x) & for\, x \in [a,b], \\ 1 & for\, x \in [b,c], \\ g(x) & for\, x \in [c,d], \\ 0 & for\, x \ge d \end{cases}$$

where $a \le b \le c \le d$, *f* is a continuous function that increases to 1 at point *b*, and *g* is a continuous function that decreases from 1 at point *c*. The most common types of fuzzy numbers are triangular and trapezoidal shapes.

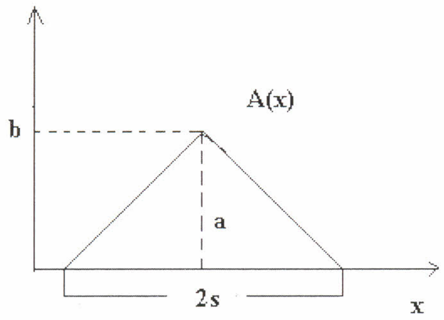


Fig. 1. Triangular shape fuzzy number

Any symmetric, triangular-shaped membership function which is characterized by the three parameters, *a, b,* and *s*, as shown in the Fig. 1, is represented by the generic form:

$$A(x)=\begin{cases} b\left(1-\dfrac{|x-a|}{s}\right) & when \quad a-s \le x \le a+s \\ 0 & otherwise \end{cases}$$

Although membership functions of a great variety of shapes are possible for representing fuzzy numbers, as exemplified in Fig. 2 for the concept "around k" where k is any integer. These types of fuzzy numbers are easy to construct and manipulate. Even though the choice of the real numbers a, b, c, d in the general definition of A(x) of a fuzzy number is very important and highly dependent on the context of each application, most current applications that employ fuzzy numbers are not significantly affected by the shapes of functions f and g in A(x). Hence, it is quite natural to choose simple linear functions, represented by straight lines.

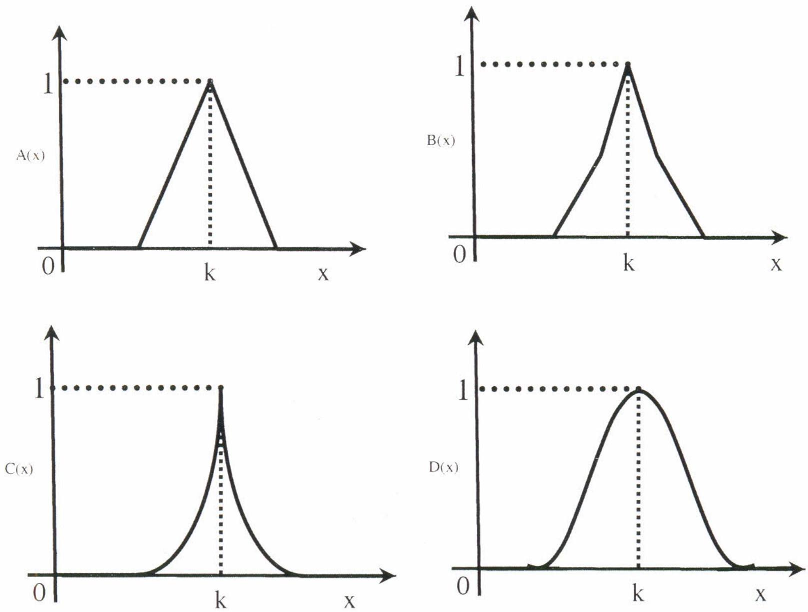


Fig. 2. Possible fuzzy numbers to capture the concept "around k".

4. DIVISION OPERATION ON INTERVALS

Let us consider two closed intervals $[a, b]$ and $[c, d]$. The endpoints of these intervals are some real numbers, denoted generically here as a, b, c and d for which $a \leq b \; and \; c \leq d$. Then for any two closed intervals of real numbers, $[a, b]$ and $[c, d]$, the result of division on these intervals is defined as

$$\frac{[a,b]}{[c,d]} = [a,b].\left[\frac{1}{d},\frac{1}{c}\right] = \left[min\left(\frac{a}{c},\frac{a}{d},\frac{b}{c},\frac{b}{d}\right), max\left(\frac{a}{c},\frac{a}{d},\frac{b}{c},\frac{b}{d}\right)\right].$$

Interval division assumes that 0 is not one of the elements in the division interval [c, d].

5. RULE OF INTERVAL MULTIPLICATION

If $A = [a_1, a_2]$ and $B = [b_1, b_2]$ then

$A \, . \, B = [a_1, a_2] \, . \, [b_1, b_2]$

$= [\min \{a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2\}, \max \{a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2\}]$ (1.1)

Depending on sign and magnitude of a_1, a_2, b_1, b_2, we get the simpler versions of Eq. (1.1).

A	B	(A . B)
$0 \le a_1 \le a_2$	$0 \le b_1 \le b_2$	$[a_1 b_1, a_2 b_2]$
$0 \le a_1 \le a_2$	$b_1 \le 0 \le b_2$	$[a_2 b_1, a_2 b_2]$
$0 \le a_1 \le a_2$	$b_1 \le b_2 \le 0$	$[a_2 b_1, a_1 b_2]$
$a_1 \le 0 \le a_2$	$0 \le b_1 \le b_2$	$[a_1 b_2, a_2 b_2]$
$a_1 \le 0 \le a_2$	$b_1 \le 0 \le b_2$	[min $\{a_1 b_2, a_2 b_1\}$, max $\{a_1 b_1, a_2 b_2\}$]
$a_1 \le 0 \le a_2$	$b_1 \le b_2 \le 0$	$[a_2 b_2, a_1 b_2]$
$a_1 \le a_2 \le 0$	$0 \le b_1 \le b_2$	$[a_1 b_2, a_2 b_1]$
$a_1 \le a_2 \le 0$	$b_1 \le 0 \le b_2$	$[a_1 b_2, a_1 b_1]$
$a_1 \le a_2 \le 0$	$b_1 \le b_2 \le 0$	$[a_2 b_2, a_1 b_1]$

6. DIVISION OPERATION ON FUZZY NUMBERS

Let us consider two fuzzy numbers U = < U^-, U^+> and V = < V^-, V^+> are defined in the standard way, in terms of the α-cuts for all $\alpha \in [0, 1]$ then if $0 \notin [V_0^-, V_0^+]$ then

$$\left(\frac{U}{V}\right)_\alpha = \left(\left(\frac{U}{V}\right)_\alpha^-, \left(\frac{U}{V}\right)_\alpha^+\right), \text{ where } \forall \alpha \in [0,1] \text{ and}$$

$$\left(\frac{U}{V}\right)_\alpha^- = \min\left\{\frac{U_\alpha^-}{V_\alpha^-}, \frac{U_\alpha^-}{V_\alpha^+}, \frac{U_\alpha^+}{V_\alpha^-}, \frac{U_\alpha^+}{V_\alpha^+}\right\} \text{ and}$$

$$\left(\frac{U}{V}\right)_\alpha^+ = \max\left\{\frac{U_\alpha^-}{V_\alpha^-}, \frac{U_\alpha^-}{V_\alpha^+}, \frac{U_\alpha^+}{V_\alpha^-}, \frac{U_\alpha^+}{V_\alpha^+}\right\}.$$

Example:
We consider two fuzzy numbers A and B. After applying division operation, we see that μ looses its triangular shape!

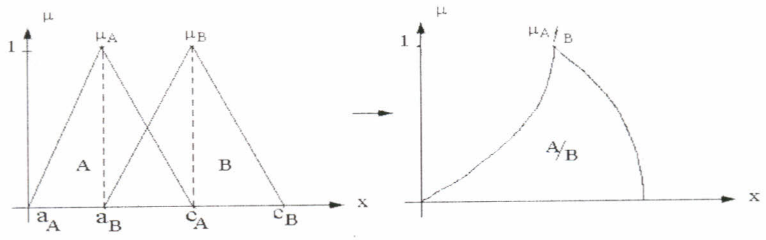


Fig.3. (a) Fuzzy Numbers, (b) Division of two Fuzzy Numbers

7. ALGORITHM

Step 1: Input fuzzy numbers, say, A(x) and B(x),

Step 2: Determine the α-cut of the given fuzzy numbers,

Step 3: Determine the values of α which lies between (0, 1]. In this case, there are two possibilities:

Case 1: There exist no α that lies between (0, 1].

Case 2: There exist unique α that lies between (0, 1].

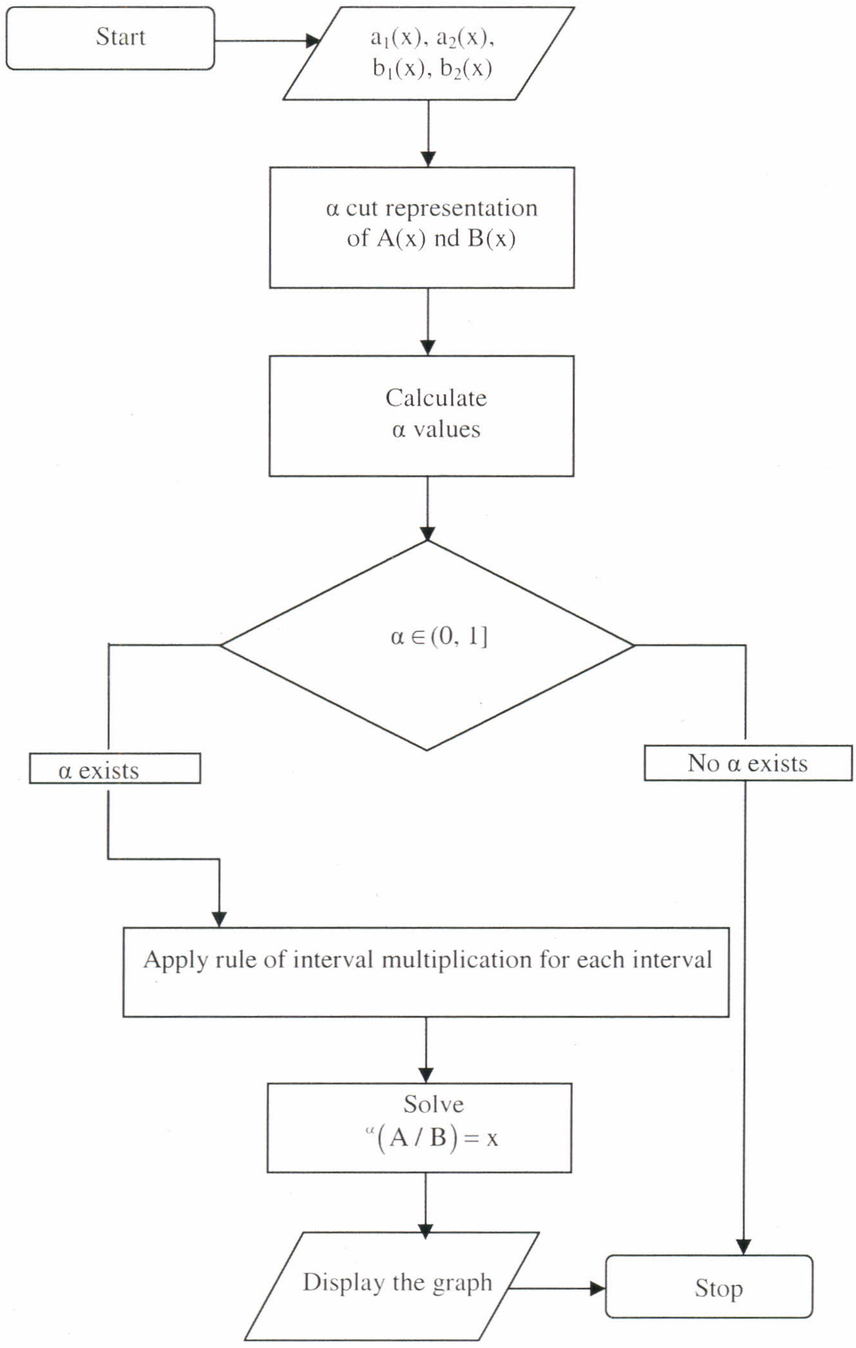
Step 4: Subdivide the interval (0, 1] depending on the values of α.

Step 5: Applying the rule of interval multiplication for each of the subinterval, we get the α-cut representation of $^{\alpha}\left(\frac{A}{B}\right)$.

Step 6: Solve the α-cut representation of $^{\alpha}\left(\frac{A}{B}\right)$ with respect to x, we get the required expressions which are denoted by (A / B) (x).

Step 7: Draw the graph of (A / B) (x).

8. FLOWCHART


Start
a1(x), a2(x),
b1(x), b2(x)
α cut representation
of A(x) nd B(x)
Calculate
α values
α ∈ (0, 1]
α exists
No α exists
Apply rule of interval multiplication for each interval
Solve
α(A / B) = x
Display the graph
Stop

9. SOME EXAMPLES

Example 1:

$$A(x) = \begin{cases} 0 & for\ otherwise \\ \frac{2+x}{2} & for\ -2 \le x \le 0 \\ \frac{2-x}{2} & for\ 0 \le x \le 2 \end{cases} \quad \text{and} \quad B(x) = \begin{cases} 0 & for\ otherwise \\ \frac{x-2}{2} & for\ 2 \le x \le 4 \\ \frac{6-x}{2} & for\ 4 \le x \le 6 \end{cases}$$

Example 2:

$$A(x) = \begin{cases} 0 & for\ otherwise \\ \frac{x+1}{2} & for\ -1 \le x \le 1 \\ \frac{3-x}{2} & for\ 1 \le x \le 3 \end{cases} \quad \text{and} \quad B(x) = \begin{cases} 0 & \text{for otherwise} \\ \frac{x-1}{2} & \text{for } 1 \le x \le 3 \\ \frac{5-x}{2} & \text{for } 3 \le x \le 5 \end{cases}$$

10. MATHEMATICA CODE

Let us consider two triangular shape fuzzy numbers

$$A(x) = \begin{cases} 0, & \text{otherwise,} \\ a_1(x), a \le x \le b, \\ a_2(x), b \le x \le c. \end{cases} \quad \text{and} \quad B(x) = \begin{cases} 0, & \text{otherwise,} \\ b_1(x), q \le x \le r, \\ b_2(x), r \le x \le s. \end{cases}$$

Input (1):

This query indicates the input of two triangular shape fuzzy numbers A(x) and B(x).

```
f[x_] := a1[x]
f1[x_] := a2[x]
g[x_] := b1[x]
g1[x_] := b2[x]
```

Input (2):

This query indicates the α-cut of A(x) and B(x).

```
z1 = Solve[f[x] == α, x];
z2 = Solve[f1[x] == α, x];
z3 = Solve[g[x] == α, x];
z4 = Solve[g1[x] == α, x];
```

Input (3):

This query gives the different values of α. If there exists no α found that lies between (0, 1] then go to Input (4). If there exist unique α value found that lies between (0, 1] then go to Input (8).

```
x1 = Solve[z1[[1, 1, 2]] == 0, α];
x2 = Solve[z2[[1, 1, 2]] == 0, α];
x3 = Solve[z3[[1, 1, 2]] == 0, α];
x4 = Solve[z4[[1, 1, 2]] == 0, α];
```

Input (4):

For the interval (0, 1], this query separates the α-cut expressions of A(x) and B(x) and also calculates their values at the given range.

```
a1 = z1[[1, 1, 2]] /. {α→Random[Real, {0.1, 0.9}]};
a2 = z2[[1, 1, 2]] /. {α→Random[Real, {0.1, 0.9}]};
b1 = z3[[1, 1, 2]] /. {α→Random[Real, {0.1, 0.9}]};
b2 = z4[[1, 1, 2]] /. {α→Random[Real, {0.1, 0.9}]};
a11 = z1[[1, 1, 2]];
a12 = z2[[1, 1, 2]];
b11 = z3[[1, 1, 2]];
b12 = z4[[1, 1, 2]];
```

Input (5):

In this query, we apply the rule of interval multiplication.

```
e1 = Block[{}, in1 = 0;
   If[0 ≤ a1 ≤ a2 && 0 ≤ b1 ≤ b2, Interval[{a11 * b11, a12 * b12}],
   If[0 ≤ a1 ≤ a2 && b1 ≤ 0 ≤ b2, Interval[{a12 * b11, a12 * b12}],
   If[0 ≤ a1 ≤ a2 && b1 ≤ b2 ≤ 0, Interval[{a12 * b11, a11 * b12}],
   If[a1 ≤ 0 ≤ a2 && 0 ≤ b1 ≤ b2, Interval[{a11 * b12, a12 * b12}],
   If[a1 ≤ 0 ≤ a2 && b1 ≤ 0 ≤ b2, Interval[{Min[{a11 * b12, a12 * b11}]
  , Max[{a11 * b11, a12 * b12}]}],
   If[a1 ≤ 0 ≤ a2 && b1 ≤ b2 ≤ 0, Interval[{a12 * b12, a11 * b12}],
   If[a1 ≤ a2 ≤ 0 && 0 ≤ b1 ≤ b2, Interval[{a11 * b12, a12 * b11}],
   If[a1 ≤ a2 ≤ 0 && b1 ≤ 0 ≤ b2, Interval[{a11 * b12, a11 * b11}],
   If[a1 ≤ a2 ≤ 0 && b1 ≤ b2 ≤ 0, Interval[{a12 * b12, a11 * b11}]
   ]]]]]]]]];
```

Input (6):

This query gives us the required expressions for division of two fuzzy numbers.

```
z5 = Solve[e1[[1, 1]] == x, α];
u1 = Solve[z5[[1, 1, 2]] == Interval[{0, 1}], x];
z6 = Solve[e1[[1, 2]] == x, α];
u2 = Solve[z6[[1, 1, 2]] == Interval[{0, 1}], x];
```

Input (7):

This query display the graph of (A / B) (x).

```
Collect[z5[[1, 1, 2]] && u1[[1, 1, 2]] || z6[[1, 1, 2]]
&& u2[[1, 1, 2]] || 0 && otherwise, x]
sol1 = Plot[z5[[1, 1, 2]], {x, u1[[1, 1, 2, 1, 1]],
u1[[1, 1, 2, 1, 2]]}, DisplayFunction → Identity];
sol2 = Plot[z6[[1, 1, 2]], {x, u2[[1, 1, 2, 1, 1]],
u2[[1, 1, 2, 1, 2]]}, DisplayFunction → Identity];
Show[{sol1, sol2}, Graphics[{Text[" (A/B)[x]", {u1[[1, 1, 2, 1, 2]]
, 0.5}]}], DisplayFunction → $DisplayFunction]
```

Input (8):

This query shows that v1 contains the unique value of α that belongs to (0, 1].

```
v1 = Block [{}, in1 = 0;
   If [0 < x1[[1, 1, 2]] < 1, x1[[1, 1, 2]],
    If [0 < x2[[1, 1, 2]] < 1, x2[[1, 1, 2]],
     If [0 < x3[[1, 1, 2]] < 1, x3[[1, 1, 2]],
      If [0 < x4[[1, 1, 2]] < 1, x3[[1, 1, 2]]]]]]]];
```

Input (9):

For the subinterval (0, v1], this query separates the α-cut expressions of A(x) and B(x) and also calculates their values at the given range.

```
a1 = z1[[1, 1, 2]] /. {α → (v1 / 2) // N};
a2 = z2[[1, 1, 2]] /. {α → (v1 / 2) // N};
b1 = z3[[1, 1, 2]] /. {α → (v1 / 2) // N};
b2 = z4[[1, 1, 2]] /. {α → (v1 / 2) // N};
a11 = z1[[1, 1, 2]];
a12 = z2[[1, 1, 2]];
b11 = z3[[1, 1, 2]];
b12 = z4[[1, 1, 2]];
```

Input (10):

In this query, we apply the rule of interval multiplication.

```
e1 = Block[{}, in1 = 0;
If[0 ≤ a1 ≤ a2 && 0 ≤ b1 ≤ b2, Interval[{a11*b11, a12*b12}],
If[0 ≤ a1 ≤ a2 && b1 ≤ 0 ≤ b2, Interval[{a12*b11, a12*b12}],
If[0 ≤ a1 ≤ a2 && b1 ≤ b2 ≤ 0, Interval[{a12*b11, a11*b12}],
If[a1 ≤ 0 ≤ a2 && 0 ≤ b1 ≤ b2, Interval[{a11*b12, a12*b12}],
If[a1 ≤ 0 ≤ a2 && b1 ≤ 0 ≤ b2, Interval[{Min[{a11*b12, a12*b11}],
Max[{a11*b11, a12*b12}]}],
If[a1 ≤ 0 ≤ a2 && b1 ≤ b2 ≤ 0, Interval[{a12*b12, a11*b12}],
If[a1 ≤ a2 ≤ 0 && 0 ≤ b1 ≤ b2, Interval[{a11*b12, a12*b11}],
If[a1 ≤ a2 ≤ 0 && b1 ≤ 0 ≤ b2, Interval[{a11*b12, a11*b11}],
If[a1 ≤ a2 ≤ 0 && b1 ≤ b2 ≤ 0, Interval[{a12*b12, a11*b11}]]]]]]]]]]];
```

Input (11):

For the subinterval (v1, 1], this query separates the α-cut expressions of A(x) and B(x) and also calculates their values at the given range.

```
a1 = z1[[1, 1, 2]] /. {α → ((v1 + 1) / 2) // N};
a2 = z2[[1, 1, 2]] /. {α → ((v1 + 1) / 2) // N};
b1 = z3[[1, 1, 2]] /. {α → ((v1 + 1) / 2) // N};
b2 = z4[[1, 1, 2]] /. {α → ((v1 + 1) / 2) // N};
a11 = z1[[1, 1, 2]];
a12 = z2[[1, 1, 2]];
b11 = z3[[1, 1, 2]];
b12 = z4[[1, 1, 2]];
```

Input (12):

In this query, we apply the rule of interval multiplication.

```
e2 = Block[{}, in2 = 0;
If[0 ≤ a1 ≤ a2 && 0 ≤ b1 ≤ b2, Interval[{a11*b11, a12*b12}],
If[0 ≤ a1 ≤ a2 && b1 ≤ 0 ≤ b2, Interval[{a12*b11, a12*b12}],
If[0 ≤ a1 ≤ a2 && b1 ≤ b2 ≤ 0, Interval[{a12*b11, a11*b12}],
If[a1 ≤ 0 ≤ a2 && 0 ≤ b1 ≤ b2, Interval[{a11*b12, a12*b12}],
If[a1 ≤ 0 ≤ a2 && b1 ≤ 0 ≤ b2, Interval[{Min[{a11*b12, a12*b11}],
Max[{a11*b11, a12*b12}]}],
If[a1 ≤ 0 ≤ a2 && b1 ≤ b2 ≤ 0, Interval[{a12*b12, a11*b12}],
If[a1 ≤ a2 ≤ 0 && 0 ≤ b1 ≤ b2, Interval[{a11*b12, a12*b11}],
If[a1 ≤ a2 ≤ 0 && b1 ≤ 0 ≤ b2, Interval[{a11*b12, a11*b11}],
If[a1 ≤ a2 ≤ 0 && b1 ≤ b2 ≤ 0, Interval[{a12*b12, a11*b11}]
]]]]]]]]]];
```

Input (13):

This query gives us the required expressions for division of two fuzzy numbers.

```
z5 = Solve[e1[[1, 1]] == x, α];
u1 = Solve[z5[[1, 1, 2]] == Interval[{0, x1[[1, 1, 2]]}], x];
z6 = Solve[e1[[1, 2]] == x, α];
u2 = Solve[z6[[1, 1, 2]] == Interval[{0, x1[[1, 1, 2]]}], x];
z7 = Solve[e2[[1, 1]] == x, α];
u3 = Solve[z7[[1, 1, 2]] == Interval[{x1[[1, 1, 2]], 1}], x];
z8 = Solve[e2[[1, 2]] == x, α];
u4 = Solve[z8[[1, 1, 2]] == Interval[{x1[[1, 1, 2]], 1}], x];
```

Input (14):

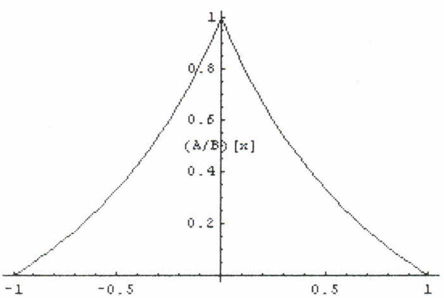
This query display the graph of $\left(\frac{A}{B}\right)(x)$.

```
Collect[z5[[1, 1, 2]] && u1[[1, 1, 2]] || z6[[1, 1, 2]] &&
u2[[1, 1, 2]] || z7[[1, 1, 2]] && u3[[1, 1, 2]] ||
z8[[1, 1, 2]] && u4[[1, 1, 2]] || 0 && otherwise, x]
sol1 = Plot[z5[[1, 1, 2]], {x, u1[[1, 1, 2, 1, 1]], u1[[1, 1, 2, 1, 2]]},
DisplayFunction → Identity];
sol2 = Plot[z6[[1, 1, 2]], {x, u2[[1, 1, 2, 1, 1]], u2[[1, 1, 2, 1, 2]]},
DisplayFunction → Identity];
sol3 = Plot[z7[[1, 1, 2]], {x, u3[[1, 1, 2, 1, 1]], u3[[1, 1, 2, 1, 2]]},
DisplayFunction → Identity];
sol4 = Plot[z8[[1, 1, 2]], {x, u4[[1, 1, 2, 1, 1]], u4[[1, 1, 2, 1, 2]]},
DisplayFunction → Identity];
Show[{sol1, sol2, sol3, sol4}, DisplayFunction → $DisplayFunction]
```

11. RESULTS

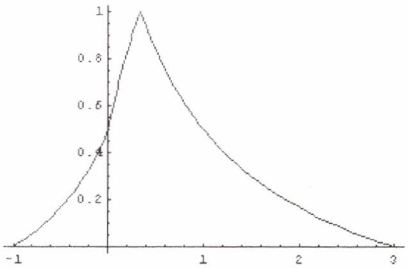
Example 1:

$$\left(\frac{A}{B}\right)(x)=\begin{cases} 0, otherwise \\ \dfrac{-1-x}{-1+x}, for -1 \le x \le 0 \\ \dfrac{1-x}{1+x}, for\ 0 \le x \le 1 \end{cases}$$



Example 2:

$$\left(\frac{A}{B}\right)(x)=\begin{cases} 0, otherwise \\ \frac{1+x}{2-2x}, for -1 \le x \le 0 \\ \frac{1+5x}{2+2x}, for\, 0 \le x \le \frac{1}{3} \\ \frac{-3+x}{-2-2x}, for \frac{1}{3} \le x \le 3 \end{cases}$$



12. CONCLUSION

We have presented a new mathematica code for evaluating the division of two fuzzy numbers using fuzzy arithmetic operations. We also use few examples to illustrate the performance evaluation process of the new mathematica code. We can see that the proposed code can efficiently handle the fuzzy arithmetic operations. We also see that the division of two fuzzy numbers does not provide triangular shape fuzzy number.

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